THE DISTRIBUTION OF BUSINESS FIRMS SIZE, STOCHASTICITIES, AND SELF-ORGANIZED CRITICALITY

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Introduction:

Decades ago, it has been noticed that the distribution of the size of business firms is "almost always highly skewed and its upper tail resembles the Pareto distribution (i.e. is a power law)".[1] This old and intriguing observation is still true today (Figure 1) and is clearly a robust pattern. Since firms engaged in very different activities contribute to build this distribution, this suggests the existence of long range effects in the economy.

Skew distributions are very common. They are found in subjects as varied as astrophysics, biology, economics, geophysics, linguistics, etc... They are sometimes referred to as 1/f distributions[2], as they seem to correspond to power laws distributions. Their origin and interpretations on the other hand are far less clear. 1/f is a signature for "Self-Organized Criticality".[3] Self-Organized Criticality suggests that the distribution has a dynamical origin.

An explanation for the distribution of business size firms, which does not have the potential to "explain" the ubiquity of power law distributions is intrinsically unsatisfactory. This criticism applies to "explanations" of the business size distribution too deeply rooted in economic assumptions.

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[1] Y. Ijiri, H. Simon, *Skew Distributions and the Size of Business Firms*, North Holland, 1977, p.138.

[2] 1/f means actually $1/f^{\alpha}$, where typically $0 < \alpha \leq 2$.

[3] P. Bak: *How Nature Works: The science of Self-Organized Criticality*, Copernicus, Springer-Verlag, 1996.

"Gibrat's presented the first formal model of the dynamics of firm size. [It is based on] the Law of Proportional Effect".[4] It states that the rate of growth of firms is constant or proportional to their size, depending of how it is defined, i.e.:

$$\frac{ds(t)}{dt} \approx \alpha s(t) \ .$$

In a stochastic context, this law can be understood as an extension of the assumption that the probability that something occurs is proportional to the number of times it has occurred in the past[5]. A stochastic argument based on this assumption can be used to derive the business firm size distribution.[6] But the empirical evidence is that the relation between rate of growth and size does not follow Gibrat's law of proportional effect. The rate of growth seems to decrease with the size.[7]

This is sometimes interpreted as implying that the stochastic interpretation of the origin of business firm size power law distributions does not work.

This paper has a dual goal:

• first to show that the relation between firms size and growth rate could take basically any form, and it still would be possible to derive a power law distribution for the firm size, from stochastic argument. This accomplished by reformulating the derivation of H. Simon in a stochastic differential equation framework.

• Second to point out that stochastic explanations of power law distributions tend to require some fine tuning. This does not seem to provide a compelling explanation of why power law distributions are so ubiquitous and robust.

Section 1: **On the relation between stochasticity and distributions**:

Stochastic differential equations[8] are a powerful tool to study stochastic processes[9]. In the case of firm sizes s(t), it would have the following general form:

$$ds(t) = a(s,t)\,dt + b(s,t)\,dz$$

[4] J. Sutton, *Journ. Econ. Literature*, **XXXV** (1997), pp. 40-60.
[5] J. Sutton, *Journ. Econ. Literature*, **XXXV** (1997), op.cit, p.
[6] H. Simon, *Biometrika*, **52** (1955), pp. 425-440.
[7] J. Sutton, *Journ. Econ. Literature*, **XXXV** (1997), *op.cit*. p. 46.
[8] RC Merton: *Journ. Econ. Theory* **3** (1971), 373-413.
[9] S. Karlin, R Taylor, Second Course in Stochastic Processes, (1981)

a(s,t) is the "drift", i.e. the average growth rate of the size s(t). dz is an infinitesimal stochastic variable, normally distributed[10]. The variance of the stochastic process is build from the function b(s,t), the "infinitesimal variance".

The solution of stochastic differential equations are distributions of values for the variable. If $\varphi(s,t:s_0)$ is the probability density distribution of the variable s(t), at time t, knowing that s(t=0)=s_0, $\varphi(s,t:s_0)$ is solution of the Kolmogorov forward equation[11]:

$$\frac{\partial\varphi(s,t:s_0)}{\partial t} = \frac{1}{2}\frac{\partial^2\left[b^2(s,t)\varphi(s,t:s_0)\right]}{\partial s^2} - \frac{\partial\left[a(s,t)\varphi(s,t:s_0)\right]}{\partial s}$$

The distribution empirically observed is the solution $\phi_\infty(s)$ of the stationary form of the Kolmogorov forward equation:

$$\frac{1}{2}\frac{\partial^2\left[b^2(s)\phi_\infty(s)\right]}{\partial s^2} - \frac{\partial\left[a(s)\phi_\infty(s)\right]}{\partial s} = 0$$

Before discussing the form of $\phi_\infty(s)$ in general, it is worth noting what happens in some very well known cases, like the Brownian motion. Brownian motion corresponds to the case where $a(s,t) = \mu$ and $b(s,t) = \sigma$. $\varphi(s,t:s_0)$ is a Gaussian, but $\phi_\infty(s)$ does not exist! The time-dependent probability distribution of s(t), $\varphi(s,t:s_0)$ is[12]:

$$\varphi(s,t:s_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{\left\{-\frac{(s-s_0-\mu t)}{\sigma^2 t}\right\}}$$

I.e. the variance increases with time. The solution $\phi_\infty(s)$ of the stationary equation can also be a Gaussian. But this happens in another case: the so-called Ornstein-Ullenbeck process ($a(s,t) = -\alpha s$, $b(s,t) = \sigma^2$)[13] or "mean-reverting" process.

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[10] Cf A Dixit, R Pindyck, Investment Under Uncertainty, Princeton Univ Press, 1993, p. 64-65.
[11] S. Karlin, R Taylor, *Second Course in Stochastic Processes*, (1981), op.cit., Eq. 5.29, p. 220.
[12] S. Karlin, R Taylor, *Second Course in Stochastic Processes*, (1981), op.cit. Eq.5.16, 5.18, p.217.
[13] S. Karlin, R Taylor, *Second Course in Stochastic Processes*, (1981), op.cit. p.221, eq. 5.35.

The stationary distribution depends on the two functions: the drift a(s) and the infinitesimal stochasticity b(s). Gibrat's law[14] corresponds to $a(s,t) \approx s$. Depending on the stochasticity b(s,t) the resulting distribution can be very different.

The general form of the solution of the stationary equation (cf Appendix A for a derivation):

$$\phi_\infty(s) = \frac{C_1}{\eta(s) b^2(s)} \int^s \eta(\xi) d\xi + \frac{C_2}{\eta(s) b^2(s)}$$

With:

$$\eta(s) = e^{-2\int^s \frac{a(y)}{b^2(y)} dy}$$

C_1 and C_2 are constants of integration. Of particular interest for this paper is to identify the situations where the stationary distribution $\phi_\infty(s)$ is a power law, i.e. where: $\phi_\infty(s) \approx s^{-\lambda}$.

This happens if: $\frac{a(y)}{b^2(y)} \approx \frac{\alpha}{y}$, because that implies:

$$\eta(s) = \left(\frac{s}{s_0}\right)^{-2\alpha} \quad \text{and:} \quad \phi_\infty(s) = \left(\frac{s^{2\alpha}}{b^2(s)}\right)\left\{\frac{C_1}{(1-2\alpha)}\left[s^{1-2\alpha} - s_0^{1-2\alpha}\right] + C_2\right\}.$$

There is an infinite number of combinations of choices of functions a(s) and b(s), which ensure that the resulting distribution is a power law.

One can read this result as meaning that whatever the drift a(s) it is always possible to find a infinitesimal variance which guarantees a power law distribution. (The value of the exponent of the power law depends on the specifics of the drift and variance). But one can also read the same result as saying that given a drift, there is one limited class of variance functions b(s) which guarantees that the resulting distribution is a power law. The variance function b(s) has to be such that: $\frac{a(y)}{b^2(y)} \approx \frac{\alpha}{y}$.

Section 2: **Relating with the Yule distribution**:

[14] " A random walk on a logarithmic scale", Y. Ijiri, H. Simon, Skew distributions and the size of business firms, North Holland, 1977, p.85.

The Yule distribution was shown to be the solution of the stationary form of the following master stochastic equation which was developed in the context of the Zipf's law in linguistic[15]:

$$\langle f(i,k+1)\rangle - f(i,k) = \left(\frac{1-\alpha}{k}\right)\{(i-1)f(i-1,k) - if(i,k)\},\ \text{i=2,k+1}$$

f(i,k) is the number of words which appeared i times in a text of k words. The equation reads: the expected number of words which appeared i times in a text of k+1 words is:

- the number of words which had appeared i times in the previous k words,
- plus the number of words which appeared i-1 times in the previous k words, multiplied by the probability that the new word is one of them
- minus the number of words which appeared i times in the previous k words, multiplied by the probability that the new word is one of them, so now it belongs to the family of the words which appeared i+1 times.

α is the probability that the new word has not appeared yet. It is assumed constant.

This master equation assumes that the probability that a word appears again is proportional to the number of times it appeared before. It is the linguistic version of Gibrat's law of proportional effect.

The stationary solution $f^*(s)$ is the solution of the master equation with $\frac{f(x,k)}{k} = \frac{f(x,k-1)}{k-1}$. The stationary solution is the Yule distribution, namely: $f^*(s) \approx B(s,\rho+1)$.

$$B(s,\rho+1) = \int_0^1 \lambda^{s-1}(1-\lambda)^{\rho} d\lambda = \frac{\Gamma(s)\Gamma(\rho+1)}{\Gamma(s+\rho+1)}$$ is the Euler Beta function.

Asymptotically this distribution follows a power law (cf Figure 2): $B(s,\rho+1) \xrightarrow[s\to\infty]{} \Gamma(\rho+1)s^{-(\rho+1)}$.

Implicit in the derivation of the Yule distribution from the master equation, is a special form for the stochasticity or variance function.

[15] H. Simon, *Biometrika*, **52** (1955), op.cit., eq.2.1 p. 427. If the words appearing in a text are rank ordered according to their frequency, the Zipf's law states that the rank order of a word is inversely proportional to their frequency.

In order to find which stochastic process has as Kolmogorov forward equation, the master equation for f(i,k), we first notice that f(i,k) is related to the probability distribution $\varphi(x,t)$ that some words have appeared x times by: $\varphi(x,t) = \frac{xf(x,k)}{k}$.
I.e. $\varphi(x,t)$ is the ratio between the number of words which appeared x times divided by the total number of words.

The discrete time version of the forward Kolmogorov equation can be written:

$$\frac{\partial\varphi(x,t)}{\partial t} = x\left[\frac{f(x,k)}{k} - \frac{f(x,k-1)}{k-1}\right] =$$
$$\frac{1}{2k}\left\{\begin{bmatrix} b^2(x+1)(x+1)f(x+1,k) - 2b^2(x)xf(x,k) + \\ b^2(x-1)(x-1)f(x-1,k) - [a(x)xf(x,k) - a(x-1)(x-1)f(x-1,k)]\end{bmatrix}\right\}$$

Notice that the condition for steady state ($\frac{f(x,k)}{k} = \frac{f(x,k-1)}{k-1}$) is identical to the one used by H. Simon[16]. The steady state solution f*(x) is solution of:

$$\begin{aligned}&\left[b^2(x+1)(x+1)f^*(x+1) - 2b^2(x)xf^*(x) + b^2(x-1)(x-1)f^*(x-1)\right] = \\ &\quad\left[a(x)xf^*(x) - a(x-1)(x-1)f^*(x-1)\right]\end{aligned} \qquad (1)$$

Assuming $a(x) \propto x$, the solution of the stationary form of the master equation is[17]:

$$f^*(x) = B(x,\rho+1)f^*(1)$$

with:

$$\rho = \frac{1}{1-\alpha}$$

Implicit in this derivation is an assumption about the nature of the stochasticity b(x). Assuming we know that $a(x) = x$ and $f^*(x) = B(x,\rho+1)f^*(1)$,

The continuous time equivalent of eq. 1 is::

$$\frac{1}{2}\frac{\partial^2\left[b^2(x)xf^*(x)\right]}{\partial x^2} = \frac{\partial\left[a(x)xf^*(x)\right]}{\partial x}$$

[16] H. Simon, *Biometrika*, **52** (1955), op.cit., eq. 2.8 p. 428.
[17] H. Simon, *Biometrika*, **52** (1955), op.cit., Eq 2.13, p. 429.

using this equation, b(x) is the solution of:

$$\frac{\partial\left[b^2(x)xB(x,\rho+1)\right]}{\partial x} = 2x^2B(x,\rho+1)+C_1$$

I.e. it is:

$$b(x)=\left[\frac{1}{xB(x,\rho+1)}\left\{2\int_0^x y^2B(y,\rho+1)\,dy+C_1x+C_2\right\}\right]^{1/2}$$

It is easy to show (Appendix C and Figure 3) that for large x: $b(x)\approx x$. This means that asymptotically the stochastic process behaves like a "geometric Brownian motion", i.e. it becomes similar to the solution of a stochastic differential equation of the general form: $dx=\mu\,xdt+\sigma\,xdz$. It is identical to: $d\ln x=\mu\,dt+\sigma\,dz$, i.e., " a random walk on a logarithmic scale", like Gibrat's law[18].

Section 3: **Firms size distribution**:

The following stochastic differential equation for the firm size s(t):

$$ds(t)=a(s,t)\,dt+b(s,t)\,dz$$

assumes that the evolution of the size of firms is stochastically determined from their present size. There is no other exogenous factors. Using the arguments of the previous sections, we get that the assumption $\frac{a(y)}{b^2(y)}=\frac{\lambda}{2y}$ leads to the family of power law distributions:

$$\phi_\infty(s)\approx\frac{C_1s}{b^2(s)(1-\lambda)}+\frac{C_2s^\lambda}{b^2(s)}$$

Assuming: $b(s)\approx s^?$, in obvious notations, the general form of the distribution becomes:

[18] Y. Ijiri, H. Simon, Skew distributions and the size of business firms, North Holland, 1977, p.85.

$$\phi_\infty(s) \approx C'_1 s^{1-2\gamma} + C'_2 s^{\lambda-2\gamma}$$

C'_1 and C'_2 are integration constants. $C'_1 \neq 0$ is incompatible with the results of the previous section. Assuming therefore C'_1=0, one gets:

$$\phi_\infty(s) = \frac{C_2}{\eta(s)b^2(s)} \approx s^{\lambda-2\gamma}.$$

Gibrat's laws means $a(s) \approx s$. In order to get a power law distribution for the business firm, one needs: $\gamma = 1$, implying: $\phi_\infty(s) \approx s^{\lambda-2}$.

λ is related to ρ appearing in the previous section by: $\phi_\infty(s) \approx s^{\lambda-2} \approx s^{-\rho}$, namely: $\lambda = 2-\rho$. $\rho = \dfrac{1}{1-\alpha}$. $\alpha \geq 0$ is the probability that a new word appears. It implies $\rho > 1$ or here: $\lambda < 1$.

More generally, we have shown that power law distributions occurs in the class of stochastic processes described by the stochastic differential equation[19]:

$$ds(t) = c(s,t)\left\{\lambda\, dt + \sqrt{\frac{s}{c(s,t)}}dz\right\}$$

where c(s,t) is such that.: $c(s,t) = s^{-\tau} f(t)$, where f(t) can be a function of t.[20]

This implies: $\phi_\infty(s) \approx C'_2 s^{\lambda-1+2\tau}$. Which in turn implies (cf Figure 1) that: $\lambda - 1 + 2\tau \approx -0.6$, or, assuming that the average rate of growth of firm $\lambda \approx 0.03$, we get: $\tau \approx 0.18$.

This means that in order to derive the exponent of the power law observed for the firm size distribution, starting from a stochastic differential equation, the assumption that the average growth of size is 3% leads to the requirement that the size dependence of the growth goes like $s^{-0.18}$. This is not incompatible with the empirical evidence[21].

What this result suggests is that it is possible to build a consistent argument to explain the power law or Paretian character of the size distribution, by assuming that the sizes obey a stochastic differential equation.

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[19] This implies: $\dfrac{a(s,t)}{b(s,t)^2} = \dfrac{\lambda}{s}$, $\dfrac{1}{b(s,t)^2} = \dfrac{1}{sc(s,t)}$ and: $\phi_\infty(s) \approx C'_2 s^{\lambda-1+2\tau}$

[20] $a(s,t)$ and $b(s,t)$ do not seem to be constant, but slowly decreasing function with time while the firms still follow the same distribution.

[21] D Evans, *Journ. Ind. Econ*. **XXXV** (1987), pp. 567-581.

But the values of the parameters are severely constrained. This kind of stochastic argument provides a consistent picture but fails to give an intuitive characterization of those processes which has the potential to explain their apparent universality.

Section 4: **Business firms size distribution as evidence of the self-organized character of the economy:**

The distribution of size of firms is an aspect of the organization of the economy. This distribution expresses a dynamical equilibrium between dynamical objects. The growth of firms is known to be correlated to age and size of the firm. The correlation with age is more an anti correlation[22]. The growth firm relationship is "highly non-linear", i.e. it depends on the size of the firm. As we saw in the previous section this does not preclude the possibility that the dynamics driving the size distribution be stochastic.

The power law form of the distribution suggests long range effects in the economy that presumably transcend the relation between size, age and growth of firms. The specifics of the distribution are probably related to those relations. But the ubiquity of such distributions suggests that it is a robust dynamical pattern, whose origin is still to elucidate.

Power law distributions $\varphi(x) \propto x^{-\alpha}$ satisfy: $\varphi(\mu x) = \mu^{H}\varphi(x)$. They are self-similar, α-stable[23] or "stable Paretian"[24] distributions with index of self-similarity H=-α. [25]

• What is the dynamical origin of the spontaneous emergence of self-similar distributions for business sizes? Is it possible to recognize some features in common with other dynamical systems?

• The parameter in α in "α-stable" plays a central role for the distribution. Can one find an interpretation to its value in the economy?

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[22] D. S. Evans, *Journ. Pol. Econ*. **95** (1987), pp.657-674.

[23] G. Samorodnitsky, MS Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman Hall, New York, (1994), op.cit. p. 9-11.

[24] B Mandelbrot, *The Journ. of Business*, (1963) pp. 394-420.

[25] G. Samorodnitsky, MS Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman Hall, New York, (1994), p. 309.

Before speculating about the first question, we address the second. Another way to put it which may go a bit further is to use the result that an α-stable distribution is the fixed point (or asymptotic distribution) of the renormalization group transformation[26]:

$$(T_n X)_i = \frac{1}{n^{\delta}} \sum_{j=in}^{(i+1)n-1} X_j \quad \text{with } \delta = \frac{1}{\lambda}$$

$\{X_j\}$ here represents the firm sizes, assumed to build an α-stable distribution. n is the number of firms randomly put together to build a set of new firms. The renormalization group can be construed as a rescaling. Invariance under the renormalization group is therefore a scale invariance. $\{(T_n X)_i\}$ builds the same α-stable distribution as $\{X_j\}$. If the firms were randomly aggregated n by n, the new set of firms would build the same distribution, provided the elements were divided by $n^{\frac{1}{\lambda}}$ where λ is the slope of the distribution in a log-log plot.

This statement is strictly true only for infinite α-stable distributions. Suppose one take some license from this constraint and let n go to the limit $n = n_{total}$=total number of firms. At the limit the distribution is reduced to one element which represents the total size of all the firms put together, i.e. the total work force L. The denominator becomes $n_{total}^{\frac{1}{\lambda}}$.

If one assumes that the total output Y is proportional to the total number of firms, one gets $Y \propto L^{\lambda}$ as a consequence of the α-stable character of the size distribution of firms.

This means that the exponent α appearing in the neo-classical expression for the production: $Y = L^{\alpha} f(K,T)$ is the same as the slope of the firm size distribution. This is compatible with empirical data[27].

In the area of phase transitions[28], the exponent α or λ appearing in the renormalization group equations, corresponds to a "critical" exponent. This reinforces the impression that the origin of the α-stable character of the firm size distribution has a dynamical origin reflecting the competition between firms and involving entry, exit, mergers among other things.

The fact that the size of business firms fall in 1/f distributions is clearly a robust phenomenon which suggests that the sizes of firms involved in completely unrelated

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[26] G. Samorodnitsky, MS Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, Chapman Hall, New York, (1994), op.cit. p. 4.

[27] MJ Boskin, LJ Lau in edited book >1990!

[28] ref. for phase transition

areas are correlated. Firms do not compete with all the other firms to the same extent and at the same time.

The emergence of 1/f distributions suggests that the dynamics building the distribution of sizes, act like local interactions (involving only a few firms at time) generating to long range forces in the economy coupling apparently unrelated areas, as happens in phase transitions[29] and self-organized criticality[30]. It has been recently observed that this also occurs in a class of finite size systems[31].

Inspired by this example, we study the following system:

$$s_i(t+1) = \left(s_i(t) + \lambda s_i(t)^\rho\right) - g\left(s_{i+1}(t)\right)$$

The size at t+1 is related to the size of the same firm at time t, and to the size of the nearest neighbor through the coupling $g\left(s_{i+1}(t)\right)$ (local interaction). We tried two kinds of coupling: linear:

$g \cdot s_{i+1}(t)$ and logistic: $s_i(t+1) = \left(s_i(t) + \lambda s_i(t)^\rho\right) - g \cdot s_{i+1}(t)\left(1 - \frac{s_{i+1}(t)}{s_{\max}}\right)$. In both cases we assume that when the size of the company becomes negative the firm exits. And we allow new entries with a variety of rules involving the choice of a random values for the size of the new entrants. We find a power law distribution for a large variety of cases (cf Figure 3a and 3b). The slope of the distribution depends on the values of the parameters λ , ρ and the specifics of g. The size of the fluctuations around the power laws depend on the number of firms, and the values of the parameters. But the appearance of the distribution seems a robust phenomenon.

In this system which involves only local non-linear interactions, long range forces seem to be generated. An attractive (because natural) hypothesis is that such relations are a natural manifestation of ecological interactions between firms.

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[29] H. Chaté, A Lemaître, P Marcq, P Manneville, *Physica* **A 224** (1996), p.447-457.
[30] P. Bak: *How Nature Works: The science of Self-Organized Criticality*, Copernicus, Springer-Verlag, 1996, op.cit.
[31] P Marcq, H. Chaté, P Manneville, *Phys. Rev*. **E55** (1997), pp. 2606-2627.

Appendix A: Solving the Kolmogorov forward equation:

$$\frac{\partial\left[b^2(y)\psi(y)\right]}{\partial y} = 2a(y)\psi(y) + C_1, \quad \text{where } C_1 \text{ is an integration constant.}$$

$$\frac{\partial\left[s(y)b^2(y)\psi(y)\right]}{\partial y} = C_1 s(y) \quad <=> \quad \frac{\partial\left[b^2(y)\psi(y)\right]}{\partial y} + \frac{b^2(y)\psi(y)}{s(y)}\frac{\partial[s(y)]}{\partial y} = C_1$$

I.e.:

$$\frac{b^2(y)\psi(y)}{s(y)}\frac{\partial[s(y)]}{\partial y} = -2a(y)\psi(y)$$

Or:

$$\frac{\partial\left[\ln[s(y)]\right]}{\partial y} = -\frac{2a(y)}{b^2(y)}, \text{ implying : } s(y) = e^{-2\int^{y}\frac{a(y')}{b^2(y')}dy'}$$

Then the equation:

$$\frac{\partial\left[s(y)b^2(y)\psi(y)\right]}{\partial y} = C_1 s(y) \text{ leads to:}$$

$$\psi(y) = \frac{C_1}{s(y)b^2(y)}\int^{y} s(\xi)d\xi + \frac{C_2}{s(y)b^2(y)}$$

If one assumes: $\frac{a(y)}{b(y)^2} = \frac{1}{y\sigma^2}$, then: $s(y) = e^{-2\int^{y}\frac{a(y')}{b^2(y')}dy'} = \left(\frac{y}{y_0}\right)^{-\frac{2}{\sigma^2}}$

And:

$$\psi(y) = \frac{1}{b^2(y)}\left(\frac{y}{y_0}\right)^{\frac{2}{\sigma^2}-1}\left\{\frac{C_1 y}{\left(-\frac{2}{\sigma^2}+1\right)}\left[\left(\frac{y}{y_0}\right)^{1-\frac{2}{\sigma^2}} - 1\right] + C_2\left(\frac{y}{y_0}\right)\right\}$$

Appendix B: Yule distribution and stochasticity:

Making the change of variable $z = y / x$, on:

$$b^2(x)B(x,\rho+1) = 2x^2 \int_0^1 zB(zx,\rho+1)\,dz + C_1 x + C_2$$

Implies that when: $x \to \infty$,

$$B(x,\rho+1) \to x^{-\rho-1}$$

And:

$$x^2 \int_0^1 zB(zx,\rho+1)dz \to x^2 \int_0^1 z(zx)^{-\rho-1} dz \to x^{2-\rho-1}$$

So: $b^2(x) \to x^2$, i.e. asymptotically the stochastic process is like a geometric Brownian motion.

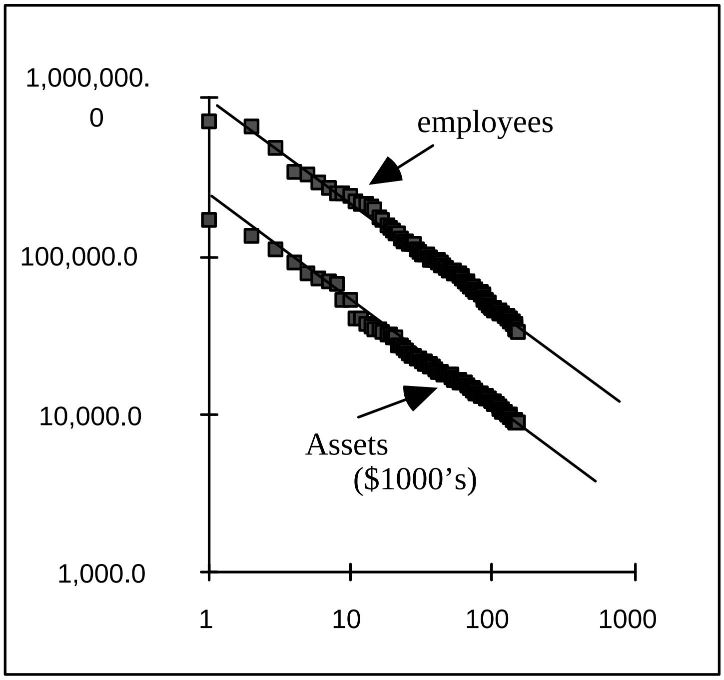


Figure 1: Log-Log plot of the size distribution of the 100 largest firms in 1995 according to the Fortune 500. Horizontally the rank-order of the company is shown. Vertically, the size as measured either in revenues (thousands of dollars), or by number of employees. The fit suggests: $s = s_{max} r^{-0.6}$. s_{max} is the "size" of the largest one, General Motors, with $168.8286 millions in revenues, in 1995. a General Motors was also the largest one by employees: 709,000 employees in 1995.

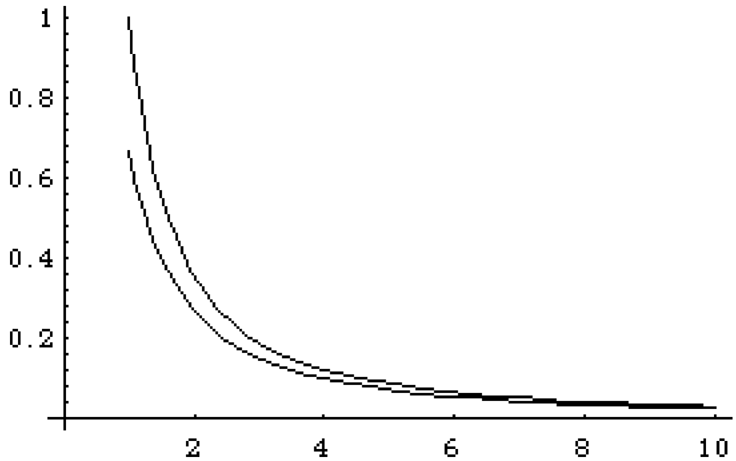


Figure 2A: Power law nature of the Yule distribution.

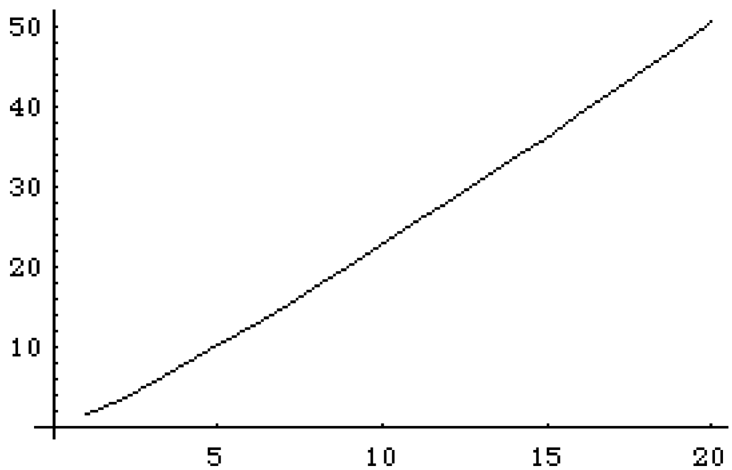


Figure 2B: implicit stochasticity b(s) in Yule distribution used in H. Simon, *Biometrika*, **52** (1955), pp. 425-440.

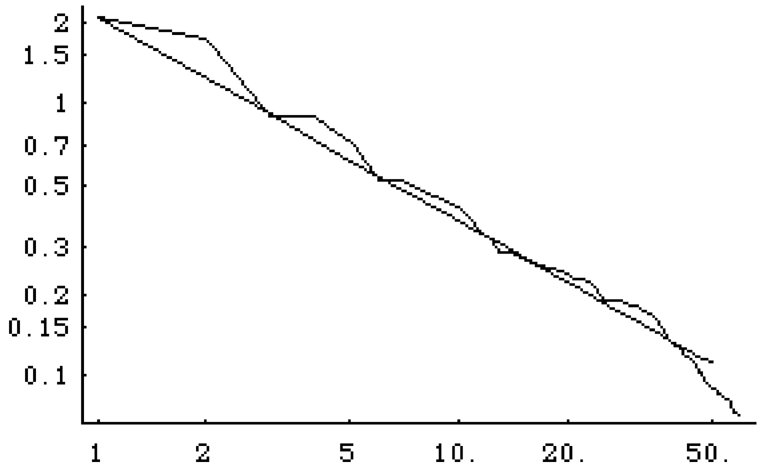


Figure 3a:1/f and nearest neighbor non-linear couplings: this curves were obtained assuming: $s_i(t+1) = \left(s_i(t) + 0.04 \times s_i(t)^{0.7}\right) - 0.4 \times \left[s_{i+1}(t) \left(1 - \frac{s_{i+1}(t)}{s_{\max}} \right) \right]$.

The speed at which the distribution of the 50 largest becomes 1/f, depends on the number of firms. The units for the size are arbitrary here.

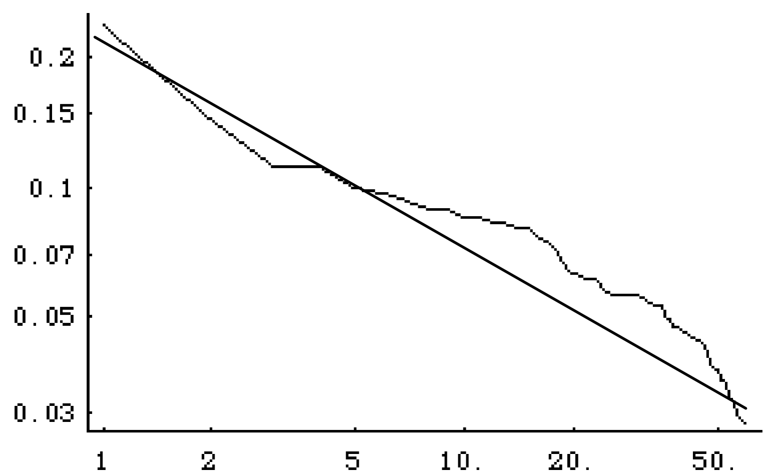


Figure 3b:1/f and nearest neighbor non-linear couplings: this curves were obtained assuming: $s_i(t+1) = \left(s_i(t) + 0.03 \times s_i(t)^{0.7}\right) - 0.05 \times s_{i+1}(t)$. The units for the size are arbitrary.

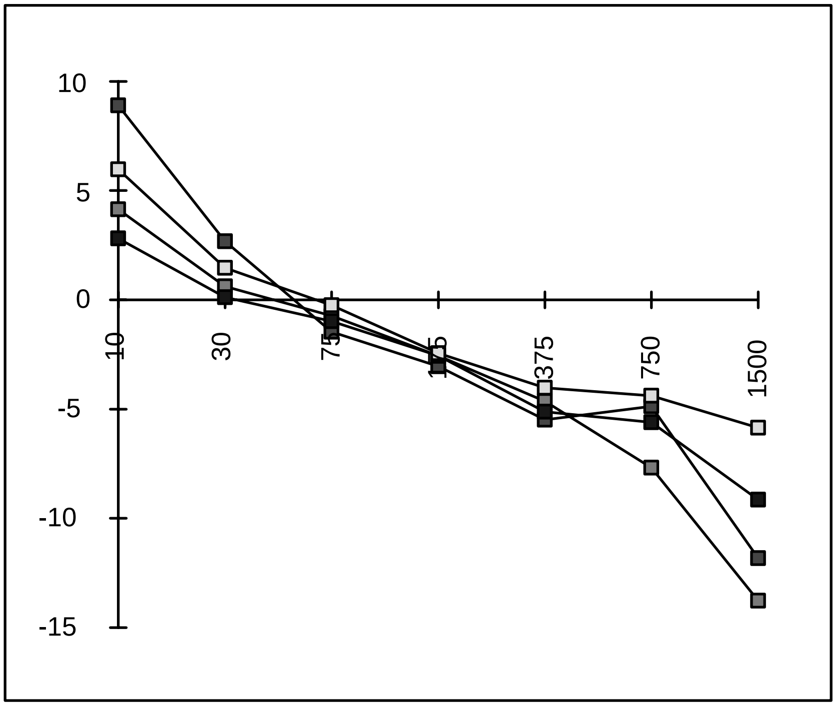
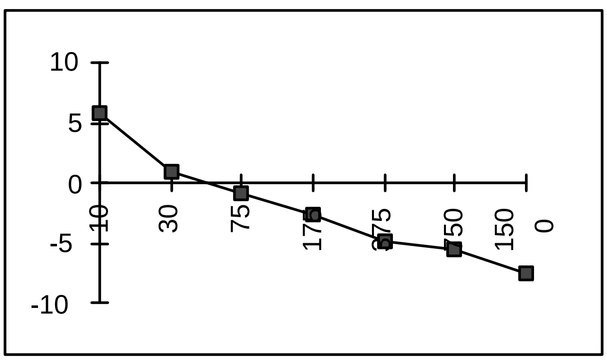


Figure 4: results of Evans for dln[s] at different ages.

The average:



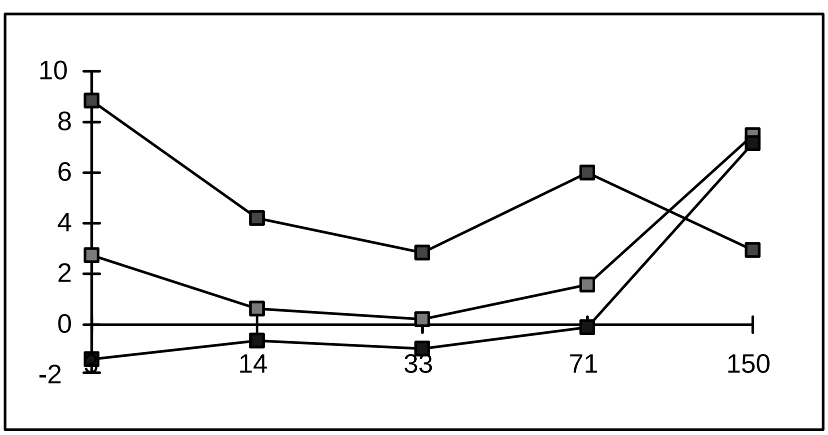


Figure 5: age effect not compelling on growth rate, more on exit rate.... Affects coupling to the environment.